Determine Equivalence Expression Using a Truth Table (cont.)

e.g. **Example:** Determine the equivalence relationship among the following three formulas:

$$p\to(q\to r),\ (p\to q)\to r,\ (p\wedge q)\to r$$

Solve:

p	q	r	$p\to(q\to r)$	$(p\to q)\to r$	$(p\wedge q)\to r$
0	0	0	1	0	1
0	0	1	1	1	1
0	1	0	1	0	1
0	1	1	1	1	1
1	0	0	1	1	1
1	0	1	1	1	1
1	1	0	0	0	0
1	1	1	1	1	1

Conclusion:

$p\to(q\to r) \Leftrightarrow (p\wedge q)\to r$

Basic Equivalence Expressions

- Double Negation Law: $\neg\neg A \Leftrightarrow A$
- Idempotent Law: $A \vee A \Leftrightarrow A$, $A \wedge A \Leftrightarrow A$
- Commutative Law: $A \vee B \Leftrightarrow B \vee A$, $A \wedge B \Leftrightarrow B \wedge A$
- Associative Law: $(A \vee B) \vee C \Leftrightarrow A \vee (B \vee C)$
  $(A \wedge B) \wedge C \Leftrightarrow A \wedge (B \wedge C)$
- Distributive Law: $A \vee (B \wedge C) \Leftrightarrow (A \vee B) \wedge (A \vee C)$
  $A \wedge (B \vee C) \Leftrightarrow (A \wedge B) \vee (A \wedge C)$
- De Morgan's Laws: $\neg(A \vee B) \Leftrightarrow \neg A \wedge \neg B$
  $\neg(A \wedge B) \Leftrightarrow \neg A \vee \neg B$
- Absorption Law: $A \vee (A \wedge B) \Leftrightarrow A$
  $A \wedge (A \vee B) \Leftrightarrow A$

↳ Basic Equivalence Expressions (cont.)

- Zero Law: $A \vee 1 \Leftrightarrow 1, \quad A \wedge 0 \Leftrightarrow 0$
- Identity Law: $A \vee 0 \Leftrightarrow A, \quad A \wedge 1 \Leftrightarrow A$
- Law of the Excluded Middle: $A \vee \neg A \Leftrightarrow 1$
- Law of Contradiction: $A \wedge \neg A \Leftrightarrow 0$
- Implication Equivalence: $A \rightarrow B \Leftrightarrow \neg A \vee B$
- Biconditional Equivalence: $A \leftrightarrow B \Leftrightarrow (A \rightarrow B) \wedge (B \rightarrow A)$
- Contraposition: $A \rightarrow B \Leftrightarrow \neg B \rightarrow \neg A$
- Negation of Equivalence: $A \leftrightarrow B \Leftrightarrow \neg A \leftrightarrow \neg B$
- Reductio ad Absurdum (Proof by Contradiction):

$$(A \rightarrow B) \wedge (A \rightarrow \neg B) \Leftrightarrow \neg A$$

↳ Equivalence Calculus • Substitution Rule

- **Equivalence Calculus:** The process of deriving new equivalences from known equivalences.
- **Substitution Rule** If $A \Leftrightarrow B$, then $\Phi(B) \Leftrightarrow \Phi(A)$

e.g. **Example:** To prove $p \to (q \to r) \Leftrightarrow (p \land q) \to r$

Proof: $p \to (q \to r)$

$\Leftrightarrow \neg p \lor (\neg q \lor r)$ (Implication Equivalence)

$\Leftrightarrow (\neg p \lor \neg q) \lor r$ (Associative Law)

$\Leftrightarrow \neg(p \land q) \lor r$ (De Morgan's Law)

$\Leftrightarrow (p \land q) \to r$ (Implication Equivalence)

↳ Methods to Prove the Non-Equivalence of Two Formulas

e.g. **Example： Proof: $p\to(q\to r) \not\Leftrightarrow (p\to q)\to r$**

- Equivalence calculus **cannot directly prove** that two formulas are not equivalent.
- The fundamental idea to prove ***non-equivalence*** is to find an assignment that makes one formula **true** while making the other **false**.
- **Key Approaches:**
  - **Truth Table Method**
  - **Observation Method**: It is easy to see that the assignment (p, q, r) = (0,0,0) makes the left formula **true** and the right formula **false**.
  - **Simplification and Observation:**

e.g. **Example：** Use **equivalence calculus** to determine the **type** of the following formula:

(1) $q \wedge \neg(p \rightarrow q)$

Solution: $q \wedge \neg(p \rightarrow q)$

$\Leftrightarrow q \wedge \neg(\neg p \vee q)$ （Implication Equivalence）

$\Leftrightarrow q \wedge (p \wedge \neg q)$ （De Morgan’s Law）

$\Leftrightarrow p \wedge (q \wedge \neg q)$ （Commutative and Associative Laws）

$\Leftrightarrow p \wedge 0$ （Law of Contradiction）

$\Leftrightarrow 0$ （Zero Law）

Thus, the formula is a ***contradiction***.

e.g. **Example:** Use **equivalence calculus** to determine the **type** of the following formula:

(2) $(p \to q) \leftrightarrow (\neg q \to \neg p)$

Solution: $(p \to q) \leftrightarrow (\neg q \to \neg p)$

$\Leftrightarrow (\neg p \vee q) \leftrightarrow (q \vee \neg p)$ （Implication Equivalence）

$\Leftrightarrow (\neg p \vee q) \leftrightarrow (\neg p \vee q)$ （Commutative Law）

$\Leftrightarrow 1$

this formula is a **tautology** (always true).

↳ To determine the type of the Formulas(e.g.)

(3) $((p \wedge q) \vee (p \wedge \neg q)) \wedge r$

Solution: $((p \wedge q) \vee (p \wedge \neg q)) \wedge r$

$\Leftrightarrow (p \wedge (q \vee \neg q)) \wedge r$ （Distributive Law)）

$\Leftrightarrow p \wedge 1 \wedge r$ （Law of Excluded Middle）

$\Leftrightarrow p \wedge r$ （Identity Law）

- This is a **satisfiable formula**, but not a **tautology**. For example:
- **101** is a **truth assignment** that makes it true.
- **000** is a **truth assignment** that makes it false.

■ **Summary:**

- A formula *A* is a ***contradiction*** if and only if A≡0.
- A formula *A* is a ***tautology*** if and only if A≡1.

- **Definition 2.12: F:{0,1}^{n}→{0,1} n-ary truth-value function.**
  - The **n** propositional variables can form 2^{2^n} truth-value functions (n≥1).
  - Each **propositional formula** corresponds to a **truth-value function.**
  - Each **truth-value function** corresponds to **infinitely many propositional formulas.**

1-ary Truth-Value Function

p	$F_0^{(1)}$	$F_1^{(1)}$	$F_2^{(1)}$	$F_3^{(1)}$
0	0	0	1	1
1	0	1	0	1

2-ary Truth-Value Function

p	q	$F_0^{(2)}$	$F_1^{(2)}$	$F_2^{(2)}$	$F_3^{(2)}$	$F_4^{(2)}$	$F_5^{(2)}$	$F_6^{(2)}$	$F_7^{(2)}$
0	0	0	0	0	0	0	0	0	0
0	1	0	0	0	0	1	1	1	1
1	0	0	0	1	1	0	0	1	1
1	1	0	1	0	1	0	1	0	1
p	q	$F_8^{(2)}$	$F_9^{(2)}$	$F_{10}^{(2)}$	$F_{11}^{(2)}$	$F_{12}^{(2)}$	$F_{13}^{(2)}$	$F_{14}^{(2)}$	$F_{15}^{(2)}$
0	0	1	1	1	1	1	1	1	1
0	1	0	0	0	0	1	1	1	1
1	0	0	0	1	1	0	0	1	1
1	1	0	1	0	1	0	1	0	1

- **Definition 2.13:** Let ***S*** be a set of logical connectives. If any **n-ary (where n≥1)** truth-value function can be represented by a formula containing only the connectives in ***S***, then ***S*** is called a **functionally complete set** (or ***complete set of connectives***).
- **Theorem 2.1:** The following sets of logical connectives are functionally complete.

why negation always include in complete set?

(1) $S_1=\{\neg, \wedge, \vee, \rightarrow, \leftrightarrow\}$

(2) $S_2=\{\neg, \wedge, \vee, \rightarrow\}$ $\quad A\leftrightarrow B \Leftrightarrow (A\rightarrow B)\wedge(B\rightarrow A)$

(3) $S_3=\{\neg, \wedge, \vee\}$ $\quad A\rightarrow B \Leftrightarrow \neg A\vee B$

(4) $S_4=\{\neg, \wedge\}$ $\quad A\vee B \Leftrightarrow \neg\neg(A\vee B) \Leftrightarrow \neg(\neg A\wedge\neg B)$

(5) $S_5=\{\neg, \vee\}$ $\quad A\wedge B \Leftrightarrow \neg(\neg A\vee\neg B)$

(6) $S_6=\{\neg, \rightarrow\}$ $\quad A\vee B \Leftrightarrow \neg(\neg A)\vee B \Leftrightarrow \neg A\rightarrow B$

↳ NAND (Not AND)&NOR (Not OR)

- **NAND Form:** $p\uparrow q \Leftrightarrow \neg(p\wedge q)$, $\uparrow$ is called the **NAND connective.**
- **NOR Form:** $p\downarrow q \Leftrightarrow \neg(p\vee q)$, $\downarrow$ is called the **NOR connective.**
  - $p\uparrow q$ is **true** if and only if p **and** q **are not both true.**
  - $p\downarrow q$ is **false** if and only if p **and** q **are not both false.**

- Theorem 2.2: $\{\uparrow\},\{\downarrow\}$ are **functionally complete sets of connectives.** (or complete set of connectives).

Proof: $\neg p \Leftrightarrow \neg(p\wedge p) \Leftrightarrow p\uparrow p$

$$p\wedge q \Leftrightarrow \neg\neg(p\wedge q) \Leftrightarrow \neg(p\uparrow q) \Leftrightarrow (p\uparrow q)\uparrow(p\uparrow q)$$

Thus, $\{\uparrow\}$ **is a functionally complete set..**

A similar proof holds for $\{\downarrow\}$.

Objective :

Key Concepts :

离散数学（011122）

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2.3 Normal Forms

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- **2.3.1 Disjunctive Normal Form (DNF) and Conjunctive Normal Form (CNF)**
  - Simple Disjunctive Form and Simple Conjunctive Form
  - Disjunctive Normal Form and Conjunctive Normal Form
- **2.3.2 Principal Disjunctive Normal Form (PDNF) and Principal Conjunctive Normal Form (PCNF)**
  - Minterms and Maxterms
  - Principal Disjunctive Normal Form and Principal Conjunctive Normal Form
  - Applications of Principal Normal Form

- **Literal**: A general term for a propositional variable and its negation.
- ***Simple Disjunctive Form***: A disjunctive formula composed of a finite number of literals.

  such as： p, $\neg q$, $p \vee \neg q$, $p \vee q \vee r$, …
- ***Simple Conjunctive Form***: A conjunctive formula composed of a finite number of literals.

  such as : p, $\neg q$, $p \wedge \neg q$, $p \wedge q \wedge r$, …
- Theorem **2.3:**

(1) A **simple disjunctive form** is a tautology if and only if it contains both a propositional variable and its negation(e.g. $p \vee \neg q$).

(2) A **simple conjunctive form** is a contradiction if and only if it contains both a propositional variable and its negation (e.g. $p \wedge \neg q$).

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- **Disjunctive Normal Form (DNF):** A disjunction composed of a finite number of simple conjunctive forms.

  $A_1 \vee A_2 \vee \ldots \vee A_r$, **where** $A_1, A_2, \ldots, A_r$ are simple conjunctive forms.

e.g. **Examples:** $p \vee q \vee \neg r$

$\neg p \wedge \neg q \wedge \neg r$

$(p_1 \wedge \neg p_2 \wedge p_3) \vee (\neg p_1 \wedge p_2 \wedge p_3) \vee (p_1 \wedge p_2 \wedge \neg p_3)$

- **Conjunctive Normal Form (CNF):** A conjunction composed of a finite number of simple disjunctive forms.

  $A_1 \wedge A_2 \wedge \ldots \wedge A_r$, **where** $A_1, A_2, \ldots, A_r$ are simple disjunctive forms.

e.g. **Examples:** $p \vee q \vee \neg r$

$\neg p \wedge \neg q \wedge r$

$(p_1 \vee \neg p_2 \vee \neg p_3) \wedge (\neg p_1 \vee p_2 \vee p_3) \wedge (p_1 \vee \neg p_3)$

- **Normal Form**: A general term referring to both disjunctive normal form (DNF) and conjunctive normal form (CNF).
- **Theorem 2.4:**

(1) A ***disjunctive normal form*** (DNF) is a contradiction if and only if each of its simple conjunctive forms is a contradiction.

(2) A ***conjunctive normal form*** (CNF) is a tautology if and only if each of its simple disjunctive forms is a tautology.

- Theorem 2.5: ***Normal Form Existence Theorem***.

  Every propositional formula has an equivalent disjunctive normal form (DNF) and conjunctive normal form (CNF).

- **Proof:** The three important steps for obtaining the normal form of a formula A

  (1) Eliminate $\rightarrow$, $\leftrightarrow$ at A

  $$A \rightarrow B \Leftrightarrow \neg A \vee B$$

  $$A \leftrightarrow B \Leftrightarrow (\neg A \vee B) \wedge (A \vee \neg B)$$

  (2) Move negation ($\neg$) inward or eliminate it:

  $$\neg\neg A \Leftrightarrow A$$

  $$\neg(A \vee B) \Leftrightarrow \neg A \wedge \neg B$$

  $$\neg(A \wedge B) \Leftrightarrow \neg A \vee \neg B$$

(3) Using the Distributive Law

$A \vee (B \wedge C) \Leftrightarrow (A \vee B) \wedge (A \vee C)$ For finding CNF

$A \wedge (B \vee C) \Leftrightarrow (A \wedge B) \vee (A \wedge C)$ For finding DNF

e.g. **Example:** **Find** $\neg(p \rightarrow q) \vee \neg r$ DNF and CNF

Solution: $\neg(p \rightarrow q) \vee \neg r$

$\Leftrightarrow \neg(\neg p \vee q) \vee \neg r$

$\Leftrightarrow (p \wedge \neg q) \vee \neg r$ DNF

$\Leftrightarrow (p \vee \neg r) \wedge (\neg q \vee \neg r)$ CNF

Note: The DNF and CNF of a formula are not unique.

Definition 2.17： minterm and maxterm

In a simple conjunctive form (or simple disjunctive form) containing ***n*** propositional variables, if each propositional variable appears exactly once in the form of a literal and the ***i-th literal*** (arranged in subscript or alphabetical order) appears in the ***i-th position*** from the left, such a simple conjunctive form (or simple disjunctive form) is called a ***minterm*** (or ***maxterm***).

e.g. **Examples:** $\neg p \wedge \neg q$， $\neg p \wedge \neg q \wedge r$ are **minterms.**

$\neg p \vee \neg q$， $\neg p \vee \neg q \vee r$ are **maxterms.**

Explanation:

(1) Propositional variables serve as placeholders in propositional logic. They do not inherently have truth values but can be replaced by specific propositions with definite truth values (1 or 0).

(2) A propositional variable can either be a simple proposition or a logical combination involving other variables.

(3) A truth table with n propositional variables, ranging from all "0" to all "1," consists of 2^n rows, corresponding to 2^n minterms and 2^n maxterms.

Explanation:

(4) The 2^n minterms (or maxterms) are all distinct from each other in terms of logical equivalence.

(5) Let m_i denote the ***i*-th minterm**, where ***i*** is the decimal representation of the truth assignment that makes the minterm true. Let M_i denote the ***i*-th maxterm**, where ***i*** is the decimal representation of the truth assignment that makes the maxterm false. m_i(or M_i) is called the name of the minterm (or maxterm).

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Minterms and Maxterms Formed by *p*,*q*

Minterm			Maxterm		
Formula	Ture	Name	Formula	False	Name
$\neg p\wedge\neg q$	0 0	m_0	$p\vee q$	0 0	M_0
$\neg p\wedge q$	0 1	m_1	$p\vee\neg q$	0 1	M_1
$p\wedge\neg q$	1 0	m_2	$\neg p\vee q$	1 0	M_2
$p\wedge q$	1 1	m_3	$\neg p\vee\neg q$	1 1	M_3

- **Theorem 2.6: Let** m_i M_i be the minterm and maxterm formed by the same set of propositional variables. Then:

$$\neg m_i \Leftrightarrow M_i, \quad \neg M_i \Leftrightarrow m_i$$

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- ***Principal Disjunctive Normal Form*** **(PDNF):** A disjunctive normal form composed of minterms.
- ***Principal Conjunctive Normal Form*** **(PCNF):** A conjunctive normal form composed of maxterms.

e.g. **Example**: n=3, propositional variables p, q, r,

$(\neg p\wedge\neg q\wedge r)\vee(\neg p\wedge q\wedge r) \Leftrightarrow m_1\vee m_3$ (PDNF)

$(p\vee q\vee\neg r)\wedge(\neg p\vee q\vee\neg r) \Leftrightarrow M_1\wedge M_5$ (PCNF)

- **Theorem 2.7:** Every propositional formula has an equivalent PDNF and PCNF, and these forms are **unique**.